

Weighted BMO spaces associated to operators

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Abstract

Let (X, d, μ) be a space of homogeneous type and $\{\mathcal{A}_t\}_{t>0}$, be a generalized approximations to the identity, for example $\{\mathcal{A}_t\}$ is a holomorphic semigroup e^{-tL} with Gaussian upper bounds generated by an operators L on $L^2(X)$. In this paper, we introduce and study the weighted BMO space $BMO_{\mathcal{A}}(X, w)$ associated to the the family $\{\mathcal{A}_t\}$. We show that for these spaces, the weighted John-Nirenberg inequality holds and we establish an interpolation theorem in scale of weighted L^p spaces. Then, we show that the dual space of the weighted Hardy space $H_L(X, w)$ associated to L as in [SY] is certain weighted BMO space $BMO_{L^*}(X, w)$ associated to the adjoint operator L^* . As applications, we prove the boundedness of two singular integrals with non-smooth kernels on the weighted BMO space $BMO_L(X, w)$.

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1 Introduction

The introduction and development of the BMO (bounded mean oscillation) function spaces on Euclidean spaces in the 1960s played an important role in modern harmonic analysis [JN, CW]. The concept of spaces of homogeneous type, which is a natural setting for the Calderón-Zygmund theory of singular integrals, was introduced in the 1970s [CW]. According to [JN], a locally integrable function f defined on \mathbb{R}^n is said to be in $BMO(\mathbb{R}^n)$, the space of functions of bounded mean oscillation, if

$$\|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f(y) - f_B| dy < \infty$$

where the supremum is taken over all balls B in \mathbb{R}^n , and f_B stands for the mean of f over B , i.e.,

$$f_B = \frac{1}{|B|} \int_B f(y) dy.$$

In [FS], Fefferman and Stein introduced the Hardy space $H^1(\mathbb{R}^n)$ and showed that the space $BMO(\mathbb{R}^n)$ is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. They also proved the characterization of functions in the BMO space by the Carleson measure. In the study of boundedness of Calderón-Zygmund operators, the Hardy space H^1 is a natural substitution to L^1 and the space BMO is a natural replacement for L^∞ . Indeed, it is well known that Calderón-Zygmund operators are bounded on L^p for $1 < p < \infty$ but bounded neither on L^1 nor on L^∞ . Meanwhile, Calderón-Zygmund operators map continuously from H^1 to L^1 and from L^∞ to BMO . Moreover, one can obtain an interpolation theorem which gives L^p boundedness from the boundedness on Hardy and on BMO spaces, i.e. if a linear operator T is bounded from H^1 to L^1 and bounded from L^∞ to BMO then by interpolation T is bounded on L^p for all $1 < p < \infty$.

In practical, there are large classes of operators whose kernels are not sufficiently smooth for them to belong to the class of Calderón-Zygmund operators. It is possible that certain operators are only bounded on L^p with the range of p being a proper subset of $(1, \infty)$. In these cases, the classical Hardy space and/or the classical BMO space are no longer suitable spaces for the study of boundedness of these operators. It is natural to raise the problem of finding new Hardy and BMO spaces which are appropriate to the study of boundedness of these operators. There has been recent extensive research in this direction. Among many works, we cite here [DY1, DY2, SY, AD2] which are closely related to this article. In [DY1, DY2], the authors studied the unweighted Hardy space H_L^1 and unweighted BMO space BMO_L associated to an operator L which satisfies the Gaussian heat kernel upper bounds and has a bounded holomorphic functional calculus. For these H_L^1 and BMO_L spaces, many important properties still hold such as the John-Nirenberg inequality, interpolation between H_L^1 and BMO_L , and the duality of BMO and Hardy spaces. Besides, it was proved that certain singular integrals with non-smooth kernels are bounded from L^∞ to BMO_L . In [SY, AD2], weighted Hardy spaces associated to operators were studied.

This paper aims to continue the line of research of [DY1, DY2, SY, AD2]. Given a family of operators $\{\mathcal{A}_t\}_{t>0}$ which is a generalized approximations to the identity (See its definition in Section 2) and a suitable weight w , we develop the theory of weighted BMO space $BMO_{\mathcal{A}}(X, w)$ associated to \mathcal{A}_t . An important example of the family $\{\mathcal{A}_t\}$ is when $\{\mathcal{A}_t\} = I - (I - e^{-tL})^M$ for some positive integer M in which e^{-tL} is a holomorphic semigroup generated by an operator L on $L^2(X)$, assuming that L satisfies Gaussian heat kernel upper bounds and has a bounded L^2 holomorphic functional calculus. Note that when the family of generalized approximations to

the identity \mathcal{A}_t is generated from L , for example $\mathcal{A}_t = I - (I - e^{-tL})^M$, we also use the notation BMO_L in place of $BMO_{\mathcal{A}}$. The new results in this article are the following:

- (i) The introduction of weighted BMO space associated to generalized approximations of identity $BMO_{\mathcal{A}}(X, w)$ (Section 3.1);
- (ii) The weighted John-Nirenberg inequality (Lemma 3.6), and the equivalence of $BMO_{\mathcal{A}}^p(X, w)$ for $1 \leq p < \infty$ (Theorem 3.5);
- (iii) The relation between w -Carleson measure and $BMO_L(X, w)$ spaces (Theorem 3.7);
- (iv) The space $BMO_{L^*}(X, w)$ is the dual space of the Hardy space $H_L(X, w)$ (Theorem 4.5) in which L^* is the adjoint operator of L and $H_L(X, w)$ was introduced in [SY] and [AD2] in the settings of Euclidean space and space of homogeneous type, respectively;
- (v) An interpolation theorem concerning $BMO_{\mathcal{A}}(X, w)$ (Theorem 5.2);
- (vi) Applications to some singular integrals with non-smooth kernels (Section 6).

We note that the approach to study Hardy spaces in [SY] relies on the geometric arguments in Euclidean setting and our approach in this paper is more in line with that of [AD2] which utilizes the tent space approach.

2 Approximations to the identity and weighted tent spaces

2.1 A Covering Lemma

Assume that (X, d, μ) is a space of homogeneous type in the sense that the set X is equipped with a metric d and a nonnegative Borel measure on X for which any ball $B(x, r)$ satisfies the doubling property

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

for any $x \in X$ and $r > 0$, where the constant $C \geq 1$ is independent of x and r . It is also required that $\mu(B(x, r)) < \infty$ for all x and r .

Note that the doubling property implies the following strong homogeneity property:

$$\mu(B(x, \lambda r)) \leq c\lambda^n \mu(B(x, r)) \quad (1)$$

for some $c, n > 0$ uniformly for all $\lambda \geq 1$ and $x \in X$. The smallest value of parameter n is a measure of the dimension of the space. There also exist c and $N, 0 \leq N \leq n$, so that

$$\mu(B(y, r)) \leq c \left(1 + \frac{d(x, y)}{r}\right)^N \mu(B(x, r)) \quad (2)$$

uniformly for all $x, y \in X$ and $r > 0$. Indeed, property (2) with $N = n$ is a direct consequence of the triangle inequality of the metric d and the strong homogeneity property. In the cases of Euclidean spaces \mathbb{R}^n and Lie groups of polynomial growth, N can be chosen to be 0.

To simplify notation, we will often use B for $B(x_B, r_B)$. Also given $\lambda > 0$, we will write λB for the λ -dilated ball, which is the ball with the same center as B and with radius $r_{\lambda B} = \lambda r_B$ and denote $V(x, r) = \mu(B(x, r))$. For each ball $B \subset X$ we set

$$S_0(B) = B \text{ and } S_j(B) = 2^j B \setminus 2^{j-1} B \text{ for } j \in \mathbb{N}.$$

Recall that the Hardy-Littlewood maximal operator M is defined by

$$Mf(x) = \sup_{B \ni x} \int_B |f(x)| d\mu(x).$$

We now give a simple covering lemma which states that we can cover a given ball by a finite overlapping family of balls with smaller radii. This will be used frequently in the sequel.

Lemma 2.1 *For any ball $B(x_B, lr)$ in X , with $l \geq 1$ and $r > 0$, then there exists a corresponding family of balls $\{B(x_{k_1}, r), \dots, B(x_{k_{N_k}}, r)\}$ such that*

(a) $B(x_{k_j}, r/3) \subset B(x_B, lr)$, for all $k = 1, \dots, N_k$;

(b) $B(x_B, lr) \subset \cup_{j=1}^{N_k} B(x_{k_j}, r)$;

(c) $N_k \leq Cl^n$;

(d) $\sum_{j=1}^{N_k} \chi_{B(x_{k_j}, r)} \leq C$, where C is independent of l and r .

Proof: By Vitali covering lemma, we can pick from the family of balls $\{B(x, \frac{r}{3}) : x \in B(x_B, lr)\}$ a disjoint family $\{B(x_{k_1}, \frac{r}{3}), \dots, B(x_{k_{N_k}}, \frac{r}{3})\}$ such that $B(x_B, lr) \subset \cup_{j=1}^{N_k} B(x_{k_j}, r)$. This shows (a) and (b).

We observe that $B(x_{k_j}, \frac{r}{3}) \subset B(x_B, 2lr)$ and $B(x_B, lr) \subset B(x_{k_j}, 6l\frac{r}{3})$ for all $j = 1, \dots, N_k$. This together with (1) gives

$$\begin{aligned} V(x_B, lr) &\geq 2^{-n} V(x_B, 2lr) \geq C \sum_{j=1}^{N_k} V(x_{k_j}, \frac{r}{3}) \\ &\geq C \sum_{j=1}^{N_k} (6l)^{-n} V(x_{k_j}, 6l\frac{r}{3}) \geq C \sum_{j=1}^{N_k} l^{-n} V(x_B, lr) = CN_k l^{-n} V(x_B, lr). \end{aligned}$$

This implies (c).

For any $x \in X$, set $I_x = \{i : x \in B(x_i, r), i \in \{k_1, \dots, k_{N_k}\}\}$. Then, $\cup_{i \in I_x} B(x_i, \frac{r}{3}) \subset \cup_{i \in I_x} B(x_i, r) \subset B(x, 2r)$. This implies

$$\begin{aligned} V(x, r) &\geq CV(x, 2r) \geq \sum_{i \in I_x} V(x_i, \frac{r}{3}) \\ &\geq C \sum_{i \in I_x} V(x_i, 4r) \geq |I_x| V(x, r). \end{aligned}$$

It therefore follows that $\sum_{j=1}^{N_k} \chi_{B(x_{k_j}, r)} \leq C$, where C is independent of l and r . Hence (d) is proved.

2.2 Approximations to the identity

We will work with a class of integral operators $\{\mathcal{A}_t\}_{t>0}$, which plays the role of generalized approximations to the identity. We assume that for each $t > 0$, the operator \mathcal{A}_t is defined by its kernel $a_t(x, y)$ in the sense that

$$\mathcal{A}_t f(x) = \int_X a_t(x, y) f(y) d\mu(y)$$

for every function $f \in \cup_{p \geq 1} L^p(X)$.

We also assume that the kernel $a_t(x, y)$ of \mathcal{A}_t satisfies the Gaussian upper bound

$$|a_t(x, y)| \leq \frac{1}{V(x, t^{1/m})} \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{t^{1/(m-1)}}\right), \quad (3)$$

for all $t > 0$ and $x, y \in X$ where m is a positive constant, $m \geq 2$.

The decay of the kernel $a_t(x, y)$ guarantees that \mathcal{A}_t is bounded on $L^p(X)$ for all $p \in (1, \infty)$. More precisely, we have the following proposition, see [DR].

Proposition 2.2 *For each $p \in [1, \infty]$, there exists a constant $c > 0$ such that for all $t > 0$,*

$$|\mathcal{A}_t f(x)| \leq \int_X \frac{1}{V(x, t^{1/m})} \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{t^{1/(m-1)}}\right) |f(y)| d\mu(y) \leq cMf(x)$$

for all $f \in L^p(X)$, μ -a.e.

2.3 Muckenhoupt weights

We first introduce some notation:

$$\oint_B h(x) d\mu(x) = \frac{1}{V(B)} \int_B h(x) d\mu(x).$$

The following presentation on Muckenhoupt weights is partly taken from [ST]. A weight w is a non-negative measurable and locally integrable function on X . We denote

$$w(E) := \int_E w(x) d\mu(x)$$

for any measurable set $E \subset X$.

For $1 \leq p \leq \infty$, let p' be the conjugate exponent of p , i.e. $1/p + 1/p' = 1$.

We say that $w \in A_p$, $1 < p < \infty$, if there exists a constant C such that for every ball $B \subset X$,

$$\left(\oint_B w(x) d\mu(x)\right) \left(\oint_B w^{-1/(p-1)}(x) d\mu(x)\right)^{p-1} \leq C.$$

For $p = 1$, we say that $w \in A_1$ if there exists a constant C such that for every ball $B \subset X$,

$$\oint_B w(x) d\mu(x) \leq Cw(x) \text{ for a.e. } x \in B.$$

The reverse Hölder classes are defined in the following way: $w \in RH_q$, $1 < q < \infty$, if there is a constant C such that for any ball $B \subset X$,

$$\left(\oint_B w^q(y) d\mu(y)\right)^{1/q} \leq C \oint_B w d\mu(x).$$

The endpoint $q = \infty$ is given by the condition: $w \in RH_\infty$ whenever, there is a constant C such that for any ball $B \subset X$,

$$w(x) \leq C \oint_B w(y) d\mu(y) \text{ for a.e. } x \in B.$$

Let $w \in A_p$, for $1 \leq p < \infty$, the weighted spaces L_w^p can be defined by

$$\left\{ f : \int_X f(x)^p w(x) d\mu(x) < \infty \right\}$$

with the norm

$$\|f\|_{L^p(w)} = \left(\int_X f(x)^p w(x) d\mu(x) \right)^{1/p}.$$

We sum up some of the properties of A_p classes in the following results, see [ST].

Lemma 2.3 *The following properties hold:*

- (i) $A_1 \subset A_p \subset A_q$ for $1 \leq p \leq q \leq \infty$.
- (ii) $RH_\infty \subset RH_q \subset RH_p$ for $1 \leq p \leq q \leq \infty$.
- (iii) If $w \in A_p$, $1 < p < \infty$, then there exists $1 < q < p$ such that $w \in A_q$.
- (iv) If $w \in RH_q$, $1 < q < \infty$, then there exists $q < p < \infty$ such that $w \in RH_p$.
- (v) $A_\infty = \cup_{1 \leq p < \infty} A_p \subset \cup_{1 < p \leq \infty} RH_p$

Lemma 2.4 *For any ball B , any measurable subset E of B and $w \in A_p$, $p \geq 1$, there exists a constant $C_1 > 0$ such that*

$$C_1 \left(\frac{V(E)}{V(B)} \right)^p \leq \frac{w(E)}{w(B)}.$$

If $w \in RH_r$, $r > 1$. Then, there exists a constant $C_2 > 0$ such that

$$\frac{w(E)}{w(B)} \leq C_2 \left(\frac{V(E)}{V(B)} \right)^{\frac{r-1}{r}}.$$

From the first inequality of Lemma 2.4, if $w \in A_1$ then there exists a constant $C > 0$ so that for any ball $B \subset X$ and $\lambda > 1$, we have

$$w(\lambda B) \leq C \lambda^n w(B).$$

2.4 Weighted tent spaces

For a measurable function F defined on $X \times (0, \infty)$, we set

$$\mathcal{S}(F)(x) = \left(\int_{\Gamma(x)} |F(y, t)|^2 \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/2}$$

and

$$\mathcal{C}(F)(x) = \sup_{B \ni x} \left(\frac{1}{w(B)} \int_{\widehat{B}} |F(y, t)|^2 \frac{V(y, t)}{w(B(y, t))} \frac{d\mu(y) dt}{t} \right)^{1/2}$$

where $\Gamma(x)$ is the cone $\{(y, t) : d(x, y) < t\}$ and $\widehat{B} = \{(y, t) : d(x, y) + t < r\}$ is the tent over B .

For $0 < p < \infty$ and $w \in L_{loc}^1(X)$, the tent space $T^p(w)$ is defined to be the set of those functions F such that $\mathcal{S}F \in L^p(w)$ and we set $\|F\|_{T^p(w)} = \|\mathcal{S}(F)\|_{L^p(w)}$, see [AD2].

When $p = \infty$, the tent space $T^\infty(w)$ is the set of all functions F such that $\mathcal{C}F \in L^\infty(X)$ and we set $\|F\|_{T^\infty(w)} = \|\mathcal{C}F\|_{L^\infty}$.

Note that the weighted tent spaces $T^p(w)$ can be considered as an extension of the tent spaces in [CMS] when $w \equiv 1$ and $X = \mathbb{R}^n$. In the case that $w \equiv 1$, we write $T^p(X)$ instead of $T^p(w)$. Another version of weighted tent spaces was investigated in [HSV] but this version is not suitable to our purpose.

A function $a(y, t)$ is said to be an $T^p(w)$ -atom whenever it is supported in \widehat{B} and

$$\left(\int_{\widehat{B}} |a(y, t)|^2 \frac{w(B(y, t))}{V(y, t)} \frac{d\mu(y)dt}{t} \right)^{1/2} \leq w(B)^{\frac{1}{2} - \frac{1}{p}}. \quad (4)$$

Let us denote $W(y, t) = \frac{w(B(y, t))}{V(y, t)}$. Then the LHS of (4) is just $\|a\|_{L^2(W)}$. It is not difficult to check that an $T^p(w)$ -atom belongs to $T^p(w)$ whenever $w \in A_1 \cap RH_2$.

We observe that if a is a $T^1(w)$ atom with supported in \widehat{B} for $w \in A_1 \cap RH_2$. Then $\frac{w(B)}{V(B)^{1/2}}$ is a T^2 -atom.

An important feature concerning weighted tent spaces is that each function in $T^p(w)$ has an atomic decomposition. More precisely, we have the following result.

Theorem 2.5 *Let $w \in A_1 \cap RH_{\frac{2}{2-p}}$ and $F \in T^p(w)$, $0 < p \leq 1$. Then there exist a sequence of $T^p(w)$ -atoms $\{a_j\}_{j \in \mathbb{N}}$ and a sequence of numbers $\{\lambda_j\}_{j \in \mathbb{N}}$ such that*

$$F = \sum_{j \in \mathbb{N}} \lambda_j a_j \quad (5)$$

and

$$\sum_{j \in \mathbb{N}} |\lambda_j|^p \leq C \|F\|_{T^p(w)}^p. \quad (6)$$

Moreover, if $F \in T^p(w) \cap T^2(X)$ then the series in (5) converges in both $T^p(w)$ and $T^2(X)$.

For the proof we refer the reader to [AD2, Theorem 3.6].

Theorem 2.6 *Assume that every ball $B(x, r)$ in (X, μ) satisfies the estimate $\mu(B(x, r)) \approx r^n$ for some positive constant n . Then for $w \in A_1 \cap RH_2$, we have*

$$\int \int_{X \times (0, \infty)} |F(y, t)G(y, t)| \frac{d\mu(y)dt}{t} \leq C \| \mathcal{C}(F) \|_{L^\infty} \| \mathcal{S}(G) \|_{L^1(w)}.$$

Moreover, the pair

$$\langle F, G \rangle \longrightarrow \int \int_{X \times (0, \infty)} |F(y, t)G(y, t)| \frac{d\mu(y)dt}{t}$$

realizes $T^\infty(w)$ as equivalent to the Banach space dual of $T^1(w)$.

Proof: In particular case when $n = 1$, the proof of this theorem was given in [HS, Lemma 5.7] and their arguments can be adapted to our situation. Hence we omit details here.

Let us denote by $T_b^p(w)$ and $T_b^p(X)$ the spaces of those functions in $T^p(w)$ and $T^p(X)$ with bounded supports, respectively. The following result will play an important role in the sequel.

Lemma 2.7 *For $w \in A_1 \cap RH_{\frac{2}{2-p}}$, the space $T_b^p(w)$ and $T_b^2(X)$ coincide for all $p \in (0, 1]$.*

For the proof, we refer the reader to [AD2, Lemma 3.8].

3 Weighted BMO spaces associated to operators

3.1 Definition of $BMO_{\mathcal{A}}(X, w)$

Throughout this paper, we assume that the family of the operators $\{\mathcal{A}_t\}_{t \geq 0}$ satisfies the Gaussian upper bounds (3) and these operators commute, i.e. $\mathcal{A}_s \mathcal{A}_t = \mathcal{A}_t \mathcal{A}_s$ for all $s, t > 0$. Note that we do not assume the semigroup property $\mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{s+t}$ on the family $\{\mathcal{A}_t\}_{t \geq 0}$.

Following [DY1], we now define the class of functions that the operators $\{\mathcal{A}_t\}_{t \geq 0}$ act upon. A function $f \in L^1_{loc}(X)$ is said to be a function of type (x_0, β) if f satisfies

$$\left(\int_X \frac{|f(x)|^2}{(1 + d(x_0, x))^{\beta} V(x_0, 1 + d(x_0, x))} d\mu(x) \right)^{1/2} \leq c < \infty. \quad (7)$$

We denote $M_{(x_0, \beta)}$ the collection of all functions of type (x_0, β) . If $f \in M_{(x_0, \beta)}$, the norm of f is defined by

$$\|f\|_{M_{x_0, \beta}} = \inf\{c : (7) \text{ holds}\}.$$

For a fixed $x_0 \in X$, one can check that $M_{(x_0, \beta)}$ is a Banach space under the norm $\|f\|_{M_{x_0, \beta}}$. For any $x_1 \in X$, $M_{(x_0, \beta)} = M_{(x_1, \beta)}$ with equivalent norms. Denote by

$$\mathcal{M} = \cup_{x_0 \in X} \cup_{0 < \beta < \infty} M_{(x_0, \beta)}.$$

Definition 3.1 A function $f \in \mathcal{M}$ is said to be in $BMO_{\mathcal{A}}(w)$ with $w \in A_{\infty}$, the space of functions of bounded mean oscillation associated to $\{\mathcal{A}_t\}_{t \geq 0}$ and w , if there exists some constant c such that for any ball B ,

$$\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})f(x)| d\mu(x) \leq c, \quad (8)$$

where $t_B = (r_B)^m$ (m is a constant in (3)) and r_B is the radius of B .

The smallest bound c for which (8) is satisfied is then taken to be the norm of f in this space and is denoted by $\|f\|_{BMO_{\mathcal{A}}(X, w)}$.

Remark: The space $BMO_{\mathcal{A}}(X, w)$, $\|\cdot\|_{BMO_{\mathcal{A}}(X, w)}$ is a seminormed vector space, with the seminorm vanishing on the space $\mathcal{K}_{\mathcal{A}}$, defined by

$$\mathcal{K}_{\mathcal{A}} = \{f \in \mathcal{M} : \mathcal{A}_t f(x) = f(x) \text{ for almost all } x \text{ and for all } t > 0\}.$$

In this paper, $BMO_{\mathcal{A}}(X, w)$ space is understood to be modulo $\mathcal{K}_{\mathcal{A}}$.

The following result gives a sufficient condition for the $BMO(X, w)$ to be contained in $BMO_{\mathcal{A}}(X, w)$. The proof for the unweighted case was given in [Ma] (see also [DY1]).

Proposition 3.2 Suppose that $w \in A_1$ and $\mathcal{A}_t(1) = 1$ for all $t > 0$, i.e., $\int_X a_t(x, y) d\mu(y) = 1$ for almost all $x \in X$. Then the inclusion $BMO(X, w) \subset BMO_{\mathcal{A}}(X, w)$ holds where

$$BMO(X, w) := \{f \in L^1_{loc} : \|f\|_{BMO(X, w)} := \sup_B \frac{1}{w(B)} \int_B |f - f_B| d\mu < \infty\}.$$

Proof: Let $f \in BMO(X, w)$. For any ball B , due to $\mathcal{A}_t(1) = 1$, we have

$$\begin{aligned}
& \frac{1}{w(B)} \int_B |f(x) - \mathcal{A}_{t_B} f(x)| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \left| f(x) - \int_X a_{t_B}(x, y) f(y) d\mu(y) \right| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \left| \int_X a_{t_B}(x, y) (f(x) - f(y)) d\mu(y) \right| d\mu(x) \\
&= \frac{1}{w(B)} \int_B \int_X \left| a_{t_B}(x, y) (f(x) - f(y)) \right| d\mu(y) d\mu(x) \\
&\leq \frac{C}{V(B)w(B)} \int_B \int_X \left| \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{t_B^{1/(m-1)}} \right) (f(x) - f(y)) \right| d\mu(y) d\mu(x) \\
&= \frac{C}{V(B)w(B)} \int_B \int_{2B} \dots d\mu(y) d\mu(x) + \sum_{j \geq 2} \frac{1}{V(B)w(B)} \int_B \int_{S_j(B)} \dots d\mu(y) d\mu(x) \\
&= I + \sum_{j \geq 2} I_j.
\end{aligned}$$

Let us estimate I first. We have

$$\begin{aligned}
I &\leq \frac{C}{V(B)w(B)} \int_B \int_{2B} |f(x) - f_B| d\mu(y) d\mu(x) + \frac{C}{V(B)w(B)} \int_B \int_{2B} |f(y) - f_{2B}| d\mu(y) d\mu(x) \\
&\quad + \frac{C}{V(B)w(B)} \int_B \int_{2B} |f_{2B} - f_B| d\mu(y) d\mu(x) \\
&\leq C \|f\|_{BMO(X, w)}.
\end{aligned}$$

For the term $I_j, j \geq 2$, we have

$$\begin{aligned}
I_j &\leq \frac{C}{V(B)w(B)} \int_B \int_{2^j B} \left| \exp(-c 2^{jm/(m-1)}) (f(x) - f(y)) \right| d\mu(y) d\mu(x) \\
&\leq \frac{C \exp(-c 2^{jm/(m-1)})}{V(B)w(B)} \left(\int_B \int_{2^j B} |f(x) - f_B| d\mu(y) d\mu(x) + \int_B \int_{2^j B} |f(y) - f_{2^j B}| d\mu(y) d\mu(x) \right. \\
&\quad \left. + \int_B \int_{2^j B} |f_{2^j B} - f_B| d\mu(y) d\mu(x) \right) \\
&\leq C(j+2)2^{-j} \|f\|_{BMO(X, w)}.
\end{aligned}$$

These estimates on I and $I_j, j \geq 2$ give $\|f\|_{BMO_{\mathcal{A}}(X, w)} \leq \|f\|_{BMO(X, w)}$. This completes our proof.

Proposition 3.3 *For $t > 0, K > 1$ and $w \in A_1$ we have for a.e. $x \in X$*

$$|(\mathcal{A}_t f(x) - \mathcal{A}_{Kt} f(x))| \leq C(1 + \log K) \frac{w(B(x, t^{1/m}))}{V(x, t^{1/m})} \|f\|_{BMO_{\mathcal{A}}(X, w)}.$$

Before coming to the proof, we would like to mention that the same estimates as in Proposition 3.3 was obtained in [DY1] under the extra assumption of semigroup property on the family $\{\mathcal{A}_t\}$. While the argument in [DY1] relies on Christ's covering lemma, our argument uses Lemma 2.1.

Proof: For any $s, t > 0$ such that $t \leq s \leq 2t$, we have

$$|\mathcal{A}_t f(x) - \mathcal{A}_s f(x)| \leq |\mathcal{A}_t((I - \mathcal{A}_s)f(x))| + |\mathcal{A}_s((I - \mathcal{A}_t)f(x))| := I_1 + I_2.$$

We first estimate I_1 . The Gaussian upper bound (3) of \mathcal{A}_t and the fact that $t \approx s$ gives that

$$\begin{aligned} I_1 &\leq \frac{C}{V(x, t^{1/m})} \int_X \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{t^{1/(m-1)}}\right) |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &\leq \frac{C}{V(x, s^{1/m})} \int_{B(x, s^{1/m})} \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{s^{1/(m-1)}}\right) |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &\quad + \sum_{j \geq 2} \frac{C}{V(x, s^{1/m})} \int_{S_j(B(x, s^{1/m}))} \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{s^{1/(m-1)}}\right) |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &= I_{11} + \sum_{j \geq 2} I_{1j}. \end{aligned}$$

For the term I_{11} , since $t \approx s$ and $w \in A_1$, we have

$$I_{11} \leq \|f\|_{BMO_{\mathcal{A}}(X, w)} \frac{w(B(x, s^{1/m}))}{V(x, s^{1/m})} \leq C \|f\|_{BMO_{\mathcal{A}}(X, w)} \frac{w(B(x, t^{1/m}))}{V(x, t^{1/m})}.$$

For $j \geq 2$, using Lemma 2.1 we can cover the annulus $S_j(B(x, s^{1/m}))$ by a finite overlapping family of at most $C2^{jn}$ balls $B(x_k^j, s^{1/m})$. Using $w \in A_1$, we can dominate the term I_{1j} as follows.

$$\begin{aligned} I_{1j} &\leq \frac{C}{V(x, s^{1/m})} \int_{S_j(B(x, s^{1/m}))} \exp\left(-c \frac{d(x, y)^{m/(m-1)}}{s^{1/(m-1)}}\right) |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &\leq \frac{C}{V(x, s^{1/m})} \int_{S_j(B(x, s^{1/m}))} e^{-c2^{j/(m-1)}} |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &\leq \sum_k \frac{C}{V(x, s^{1/m})} \int_{B(x_k^j, s^{1/m})} e^{-c2^j} |(I - \mathcal{A}_s)f(y)| d\mu(y) \\ &\leq \sum_k C \frac{w(x_k^j, s^{1/m})}{V(x, s^{1/m})} e^{-c2^{j/(m-1)}} \|f\|_{BMO_{\mathcal{A}}(X, w)} \\ &\leq \frac{w(x, 2^j s^{1/m})}{V(x, s^{1/m})} e^{-c2^{j/(m-1)}} \|f\|_{BMO_{\mathcal{A}}(X, w)} \\ &\leq C2^{jn} \frac{w(x, s^{1/m})}{V(x, s^{1/m})} e^{-c2^{j/(m-1)}} \|f\|_{BMO_{\mathcal{A}}(X, w)} \\ &\leq C2^{jn} e^{-c2^{j/(m-1)}} \|f\|_{BMO_{\mathcal{A}}(X, w)} \frac{w(x, t^{1/m})}{V(x, t^{1/m})}. \end{aligned}$$

This implies

$$I_1 \leq C \|f\|_{BMO_{\mathcal{A}}(X, w)} \frac{w(x, t^{1/m})}{V(x, t^{1/m})}.$$

A similar argument also gives

$$I_2 \leq C \|f\|_{BMO_{\mathcal{A}}(X, w)} \frac{w(x, t^{1/m})}{V(x, t^{1/m})}.$$

Therefore, we have

$$|(\mathcal{A}_t f(x) - \mathcal{A}_{t+s} f(x))| \leq C \|f\|_{BMO_A(X,w)} \frac{w(x, t^{1/m})}{V(x, t^{1/m})} \quad (9)$$

for all $t \leq s \leq 2t$.

In general case, taking $l \in \mathbb{N}$ such that $2^l \leq K < 2^{l+1}$, we can write

$$\begin{aligned} |(\mathcal{A}_t f(x) - \mathcal{A}_{Kt} f(x))| &\leq \sum_{k=1}^l |\mathcal{A}_{2^{l-k}t} f(x) - \mathcal{A}_{2^{l-k+1}t} f(x)| + |\mathcal{A}_{2^l t} f(x) - \mathcal{A}_{Kt} f(x)| \\ &\leq c \sum_{k=1}^l \|f\|_{BMO_A(X,w)} \frac{w(x, 2^{l-k}t^{1/m})}{V(x, 2^{l-k}t^{1/m})}. \end{aligned} \quad (10)$$

Since $w \in A_1$, we have

$$\frac{w(x, 2^k t^{1/m})}{V(x, 2^k t^{1/m})} \leq C \frac{w(x, t^{1/m})}{V(x, t^{1/m})}$$

for all $k \geq 0$.

This together with (10) gives

$$|(\mathcal{A}_t f(x) - \mathcal{A}_{Kt} f(x))| \leq C(1 + \log K) \|f\|_{BMO_A(X,w)} \frac{w(x, t^{1/m})}{V(x, t^{1/m})}.$$

This completes the proof.

3.2 John-Nirenberg inequality on $BMO_A(X, w)$

In this section, we will show that functions in the new class of weighted BMO spaces $BMO_A(X, w)$ satisfy the John-Nirenberg inequality. The unweighted version was obtained in [DY1]. Here, we extend to the weighted BMO spaces associated to the family of operators $\{\mathcal{A}_t\}_{t>0}$.

Definition 3.4 For $w \in A_1$ and $p \in [1, \infty)$, the function $f \in \mathcal{M}$ is said to be in $BMO_A^p(X, w)$, if there exists some constant c such that for any ball B ,

$$\left(\frac{V(B)^{p-1}}{w(B)^p} \int_B |(I - \mathcal{A}_{t_B})f(x)|^p d\mu(x) \right)^{1/p} \leq c. \quad (11)$$

where $t_B = (r_B)^m$ and r_B is the radius of B .

The smallest bound c for which (11) holds is then taken to be the norm of f in this space and is denoted by $\|f\|_{BMO_A^p(X,w)}$.

Similar to the classical case, it turns out that the spaces $BMO_A^p(X, w)$ are equivalent for all $1 \leq p < \infty$. More precisely, we have the following result.

Theorem 3.5 For $w \in A_1$ and $p \in [1, \infty)$, the spaces $BMO_A^p(X, w)$ coincide and their norms are equivalent.

Before coming to the proof Theorem 3.5 we need the following result.

Theorem 3.6 For $w \in A_1$ and $f \in BMO_{\mathcal{A}}(X, w)$, there exist positive constants c_1 and c_2 such that for any ball B and $\lambda > 0$ we have

$$w\{x \in B : |(f(x) - A_{t_B}f(x))| > \lambda\} \leq c_1 w(B) \exp\left(-\frac{c_2 \lambda V(B)}{\|f\|_{BMO_{\mathcal{A}}(X, w)} w(B)}\right). \quad (12)$$

Proof: Let us recall that if $w \in A_{\infty}$, there exist $C > 0$ and $\delta > 0$ such that for any ball B and any measurable subset $E \subset B$ we have

$$\frac{w(E)}{w(B)} \leq C \left(\frac{\mu(E)}{\mu(B)}\right)^{\delta}.$$

So, to prove (12), it suffices to show that

$$\mu\{x \in B : |f(x) - \mathcal{A}_{t_B}f(x)| > \lambda\} \leq c_1 \mu(B) \exp\left(-\frac{c_2 \lambda V(B)}{\|f\|_{BMO_{\mathcal{A}}(X, w)} w(B)}\right). \quad (13)$$

Since the proof of (13) is similar to that of Theorem 3.1 in [DY1] in which Proposition 2.6 in [DY1] is replaced by Proposition 3.3, we omit details here.

Proof of Theorem 3.5: For $f \in BMO_{\mathcal{A}}^p(X, w)$, using Hölder inequality, we have, for all balls B ,

$$\begin{aligned} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})f(x)| d\mu(x) &\leq \frac{V(B)^{1/p'}}{w(B)} \left(\int_B |(I - \mathcal{A}_{t_B})f(x)|^p d\mu(x) \right)^{1/p} \\ &\leq \left(\frac{V(B)^{p-1}}{w(B)^p} \int_B |(I - \mathcal{A}_{t_B})f(x)|^p d\mu(x) \right)^{1/p} \\ &= \|f\|_{BMO_{\mathcal{A}}^p(X, w)}. \end{aligned} \quad (14)$$

This implies that $BMO_{\mathcal{A}}^p(X, w) \subset BMO_{\mathcal{A}}(X, w)$.

Conversely, by Lemma 3.6, we have for any $f \in BMO_{\mathcal{A}}(X, w)$, the ball B and $p \in [1, \infty)$,

$$\begin{aligned} \frac{V(B)^{p-1}}{w(B)^p} \int_B |(I - \mathcal{A}_{t_B})f(x)|^p d\mu(x) &= p \frac{V(B)^{p-1}}{w(B)^p} \int_0^{\infty} \lambda^{p-1} \mu\{x \in B : |(I - \mathcal{A}_{t_B})f(x)| > \lambda\} d\lambda \\ &\leq c_p \frac{V(B)^{p-1}}{w(B)^p} \int_0^{\infty} \lambda^{p-1} w(B) \exp\left(-c_2 \frac{\lambda V(B)}{\|f\|_{BMO_{\mathcal{A}}(X, w)} w(B)}\right) d\lambda \\ &\leq c_p \|f\|_{BMO_{\mathcal{A}}(X, w)}^p. \end{aligned}$$

The proof is complete.

3.3 Characterizations of $BMO_L(X, w)$

In this section, we restrict ourself to consider the particular case when the family of operators $\{\mathcal{A}_t\}_t$ is the semigroup $\{e^{-tL}\}$ generated by a suitable operator L .

3.3.1 Assumptions on operator L

In the rest of the paper, we always assume that L is a linear operator of type ω on $L^2(X)$ with $\omega < \pi/2$; hence L generates a holomorphic semigroup e^{-zL} , $0 \leq |\operatorname{Arg}(z)| < \pi/2 - \omega$. Assume the following two conditions.

Assumption (a): The holomorphic semigroup e^{-zL} , $0 \leq |\operatorname{Arg}(z)| < \pi/2 - \omega$, is represented by the kernel $p_z(x, y)$ which satisfies the Gaussian upper bound

$$|p_z(x, y)| \leq c_\theta \frac{1}{V(x, |z|^{1/m})} \exp\left(-\frac{d(x, y)^{m/(m-1)}}{c|z|^{1/(m-1)}}\right) \quad (15)$$

for $x, y \in X$, $|\operatorname{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$.

Assumption (b): The operator L has a bounded H_∞ -calculus on $L^2(X)$. That is, there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$, and for $b \in H_\infty(S_\nu^0)$,

$$\|b(L)f\|_2 \leq c_{\nu,2} \|b\|_\infty \|f\|_2$$

for any $f \in L^2(X)$.

We now give some consequences of assumptions (a) and (b) which will be useful in the sequel.

- (i) If $\{e^{-tL}\}_{t \geq 0}$ is a bounded analytic semigroup on $L^2(X)$ whose kernel $p_t(x, y)$ satisfies the estimate (15), then for all $k \in \mathbb{N}$, the time derivatives of p_t satisfy

$$\left| t^k \frac{\partial p_t}{\partial t^k}(x, y) \right| \leq c \frac{1}{V(x, t^{1/m})} \exp\left(-\frac{d(x, y)^{m/(m-1)}}{ct^{1/(m-1)}}\right) \quad (16)$$

for all $t > 0$ and almost all $x, y \in X$. See Lemma 2.5 of [CD2].

- (ii) Under the assumptions (a) and (b), it was proved in Theorem 3.1 of [DR] and Theorem 6 of [DM] that the operator L has a bounded holomorphic functional calculus on $L^p(X)$, $1 < p < \infty$. That is, there exists $c_{\nu,p} > 0$ such that $b(L) \in \mathcal{L}(L^p, L^p)$ for $b \in H_\infty(S_\nu^0)$ and

$$\|b(L)f\|_p \leq c_{\nu,2} \|b\|_\infty \|f\|_p$$

for any $f \in L^p(X)$. For $p = 1$, the operator $b(L)$ is of weak-type $(1, 1)$.

For any $f \in \mathcal{M}$, we define

$$P_t f(x) = e^{-tL} f(x) \text{ and } Q_t f(x) = tL e^{-tL} f(x).$$

Moreover, the kernel $q_t(x, y)$ of Q_t satisfies

$$|q_t(x, y)| \leq c \frac{1}{V(x, t^{1/m})} \exp\left(-\frac{d(x, y)^{m/(m-1)}}{ct^{1/(m-1)}}\right) \quad (17)$$

This property is the same as the estimate (16) for $k = 1$.

3.3.2 Characterization of $BMO_L(X, w)$ via Carleson measure

For each integer number $M > 0$, we define

$$BMO_{L,M}(X, w) := \left\{ f \in \mathcal{M} : \sup_B \frac{1}{w(B)} \int_B |(I - P_{r_B^m})^M f(x)| dx = \|f\|_{BMO_{L,M}(X, w)} \right\}.$$

Setting $\mathcal{A}_t = I - (I - P_{r_B^m})^M$, then it is clear that \mathcal{A}_t satisfies the Gaussian upper bounds (3). So the spaces $BMO_{L,M}(X, w)$ can be considered to be the BMO spaces associated to the family $\mathcal{A}_t = I - (I - P_{r_B^m})^M$. When $M = 1$, we omit the index M to write $BMO_L(X, w)$ instead of $BMO_{L,M}(X, w)$. Note that when $M > 1$, the semigroup property on the family $\mathcal{A}_t = I - (I - P_{r_B^m})^M$ may fail. Hence, the arguments used in [DY1] may not work in our situation.

Similar to the classical result for BMO spaces, the weighted BMO spaces $BMO_{L,M}(X, w)$ can be characterized by using Carleson measure. Recall that the measure ν on $X \times (0, \infty)$ is called w -Carleson measure if for any ball B we have $\nu(\hat{B}) \leq cw(B)$.

If $(x, t) \in \hat{B}$, then $B(x, t) \subset B$. Since $w \in A_1$, we have

$$\frac{w(B)}{V(B)} \leq C \frac{w(B(x, t))}{V(x, t)} \quad \text{or equivalently,} \quad \frac{V(x, t)}{w(B(x, t))} \leq \frac{V(B)}{w(B)}.$$

These two inequalities will be used frequently in sequel.

For each $f \in BMO_{L,M}(X, w)$ and $w \in A_\infty$, we set

$$\mu_{f,w}(x, t) := |Q_{t^m}(I - P_{t^m})^M f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(y)dt}{t}.$$

The following result concerning the relation between the space $BMO_L(X, w)$ and Carleson measure.

Theorem 3.7 *Let $w \in A_1$ and $f \in BMO_{L,M}(X, w)$. Then $\mu_{f,w}(x, t)$ is a w -Carleson measure with*

$$\int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})^M f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \leq cw(B) \|f\|_{BMO_{L,M}(X, w)}^2 \quad (18)$$

for all balls $B \subset X$.

Proof: We can write

$$\begin{aligned} & \left(\frac{1}{w(B)} \int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})^M f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & \leq \left(\frac{1}{w(B)} \int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})(I - P_{r_B^m})^M f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & \quad + \left(\frac{1}{w(B)} \int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})(I - (I - P_{r_B^m})^M) f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & := I + II. \end{aligned}$$

To estimate I , we write $(I - P_{r_B^m})^M f = b_1 + b_2$ where $b_1(x) = (I - P_{r_B^m})^M f(x) \chi_{2B}(x)$. We have, by $w \in A_1$,

$$\begin{aligned} & \left(\int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})^M b_1(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & \leq c \left(\int \int_{\hat{B}} |Q_{t^m}(I - P_{t^m})^M b_1(x)|^2 \frac{V(B)}{w(B)} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & \leq \left(\frac{|B|}{w(B)} \right)^{1/2} \|\mathcal{G}b_1\|_{L^2}, \end{aligned}$$

where \mathcal{G} is the function defined by

$$\mathcal{G}f = \left(\int_0^\infty |Q_{t^m}(I - P_{t^m})^M f|^2 \frac{dt}{t} \right)^{1/2}.$$

Since L has holomorphic functional calculus on L^2 , \mathcal{G} is bounded on L^2 , see [Mc]. Hence,

$$\begin{aligned} & \left(\int \int_{\widehat{B}} |Q_{t^m}(I - P_{t^m})^M b_1|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \\ & \leq c \left(\frac{|B|}{w(B)} \right)^{1/2} \|b_1\|_{L^2(X)} \\ & = cw(B)^{1/2} \left(\frac{V(B)}{w^2(B)} \int_{2B} |(I - P_{r_B^m})^M f(x)|^2 dx \right)^{1/2}. \end{aligned}$$

At this stage, using Lemma 2.1 we can cover the ball $2B$ by the finite overlap family of at most $c2^n$ balls with radius r_B . Then, using the doubling property of w and Theorem 3.5, we have

$$\left(\int \int_{\widehat{B}} |Q_{t^m}(I - P_{t^m})b_1(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \right)^{1/2} \leq w(B)^{1/2} \|f\|_{BMO_{L,M}(X,w)}.$$

Further going, by (16) and (17) we have, for $x \in B$,

$$\begin{aligned} |Q_{t^m}(I - P_{t^m})^M b_2(x)| & \leq c \int_{X \setminus 2B} \frac{1}{V(x, t)} \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{t^{m/(m-1)}} \right) |(I - P_{r_B^m})f(y)| d\mu(y) \\ & \leq c \sum_{j \geq 2} \int_{S_j(B)} \frac{1}{V(x, r_B)} \left(\frac{r_B}{t} \right)^n \left(\frac{t}{2^j r_B} \right)^{2n+1} |(I - P_{r_B^m})f(y)| d\mu(y) \\ & \leq c \sum_{j \geq 2} \frac{w(2^j B)}{V(B)} \left(\frac{r_B}{t} \right)^n \left(\frac{t}{2^j r_B} \right)^{2n+1} \|f\|_{BMO_{L,M}(X,w)} \\ & \leq c \frac{w(B)}{V(B)} \left(\frac{t}{r_B} \right)^{n+1} \|f\|_{BMO_{L,M}(X,w)}. \end{aligned}$$

where in the last inequality we used Lemma 2.1 to pick the finite overlapping covering family of at most $C2^{jn}$ balls with radius r_B for each annulus $S_j(B)$.

Therefore, we have

$$\begin{aligned} & \int \int_{\widehat{B}} |Q_{t^m}(I - P_{t^m})^M b_2(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \\ & \leq c \|f\|_{BMO_{L,M}(X,w)}^2 \int_B \int_0^{r_B} \left(\frac{w(B)}{V(B)} \right)^2 \left(\frac{t}{r_B} \right)^{2n+2} \frac{V(B)}{w(B)} \frac{d\mu(x)dt}{t} \\ & \leq cw(B) \|f\|_{BMO_{L,M}(X,w)}^2. \end{aligned}$$

This implies $I \leq C \|f\|_{BMO_{L,M}(X,w)}$.

It remains to estimate the term II . Since

$$I - (I - P_{r_B^m})^M = \sum_{k=1}^M c_k P_{kr_B^m},$$

we have for $(x, t) \in \widehat{B}$

$$|Q_{t^m}(I - P_{t^m})^M P_{r_B^m} f(x)| = \sum_{k=1}^M c_k |Q_{t^m} P_{kr_B^m}(I - P_{t^m})^M f(x)| := \sum_{k=1}^M II_k.$$

For each k , due to $t^m + kr_B^m \approx r_B^m$, we have

$$\begin{aligned} II_k &= \left(\frac{t^m}{t^m + kr_B^m} \right) |Q_{t^m + kr_B^m}(I - P_{t^m})^M f(x)| \\ &\leq c \left(\frac{t}{r_B} \right)^m \int_X \frac{1}{V(B)} \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{(t^m + kr_B^m)^{1/(m-1)}} \right) |(I - P_{t^m})^M f(y)| d\mu(y) \\ &\leq c \left(\frac{t}{r_B} \right)^m \int_X \frac{1}{V(B)} \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{r_B^{m/(m-1)}} \right) |(I - P_{t^m})^M f(y)| d\mu(y) \\ &\leq c \left(\frac{t}{r_B} \right)^m \int_{B(x, t)} \frac{1}{V(B)} \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{r_B^{m/(m-1)}} \right) |(I - P_{t^m})^M f(y)| d\mu(y) \\ &\quad + \sum_{j \geq 2} \left(\frac{t}{r_B} \right)^m \int_{S_j(B(x, t))} \frac{1}{V(B)} \exp \left(-c \frac{d(x, y)^{m/(m-1)}}{r_B^{m/(m-1)}} \right) |(I - P_{t^m})^M f(y)| d\mu(y) \\ &:= K_1 + \sum_{j \geq 2} K_2. \end{aligned} \tag{19}$$

It is clear that

$$K_1 \leq C \left(\frac{t}{r_B} \right)^m \frac{w(B(x, t))}{V(B)} \|f\|_{BMO_{L, M}(X, w)} \leq C \left(\frac{t}{r_B} \right)^m \frac{w(B)}{V(B)} \|f\|_{BMO_{L, M}(X, w)}.$$

For $j \geq 2$, we have

$$K_j \leq \left(\frac{t}{r_B} \right)^m \int_{S_j(B(x, t))} \frac{1}{V(B)} \exp \left(-c \frac{(2^j t)^{m/(m-1)}}{r_B^{m/(m-1)}} \right) |(I - P_{t^m})^M f(y)| dy.$$

Using Lemma 2.1 to cover the annulus $S_j(B(x, t))$ by $C2^{jn}$ balls with radius t and then proceed the same argument as in the estimates I_{1j} in Proposition 3.3, we have

$$\begin{aligned} K_j &\leq C \left(\frac{t}{r_B} \right)^m \frac{w(B(x, 2^j t))}{V(B)} \exp \left(-c \frac{(2^j t)^{m/(m-1)}}{r_B^{m/(m-1)}} \right) \|f\|_{BMO_{L, M}(X, w)} \\ &\leq C \left(\frac{t}{r_B} \right)^m \frac{w(B(x, r_B))}{V(B)} \left(\frac{r_B}{2^j t} \right)^\epsilon \|f\|_{BMO_{L, M}(X, w)} \\ &\leq C 2^{-j\epsilon} \left(\frac{t}{r_B} \right)^{m-\epsilon} \frac{w(B)}{V(B)} \|f\|_{BMO_{L, M}(X, w)}. \end{aligned}$$

These estimates on K_1 and K_j for all $j \geq 2$ show that

$$|Q_{t^m}(I - P_{t^m})^M P_{r_B^m} f(x)| \leq C \left(\frac{t}{r_B} \right)^{m-\epsilon} \frac{w(B)}{V(B)} \|f\|_{BMO_{L, M}(X, w)}$$

where $\epsilon > 0$ so that $m - \epsilon > 0$.

Therefore, we have

$$\begin{aligned}
& \int \int_{\widehat{B}} |Q_{t^m}(I - P_{t^m})^M(I - (I - P_{r_B^m})^M)f(x)|^2 \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x)dt}{t} \\
& \leq c \|f\|_{BMO_{L,M}(X,w)}^2 \int_B \int_0^{r_B} \left(\frac{t}{r_B}\right)^{2m-2\epsilon} \left(\frac{w(B)}{V(B)}\right)^2 \frac{V(B)}{w(B)} \frac{d\mu(x)dt}{t} \\
& \leq cw(B) \|f\|_{BMO_{L,M}(X,w)}^2.
\end{aligned}$$

This completes our proof.

3.4 Characterization of $BMO(\mathbb{R}^n, w)$

Denote the Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. It is well-known that the kernels $p_t(x, y)$ associated to the semigroups $e^{-t\Delta}$ is given by

$$p_t(x, y) = h_t(x - y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}.$$

We define

$$\mathcal{M}_\Delta = \left\{ f \in L_{\text{loc}}^1 : \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty \right\}.$$

For $w \in A_1$, the weighted BMO spaces associated to Δ , $BMO_\Delta(\mathbb{R}^n, w)$ is defined by

$$BMO_\Delta(\mathbb{R}^n, w) := \left\{ f \in \mathcal{M}_\Delta : \|f\|_{BMO_\Delta(\mathbb{R}^n, w)} := \sup_B \frac{1}{w(B)} \int_B |f(x) - e^{-r_B^2 \Delta} f(x)| dx < \infty \right\}.$$

Theorem 3.8 *The spaces $BMO(\mathbb{R}^n, w)$ and $BMO_\Delta(\mathbb{R}^n, w)$ coincide and their norm are equivalent.*

Proof: It is well-know that $\int_{\mathbb{R}^n} h_t(x - y) dy = 1$, hence by Proposition 3.2, one has

$$BMO(\mathbb{R}^n, w) \subset BMO_\Delta(\mathbb{R}^n, w).$$

Conversely, for $f \in BMO_\Delta(\mathbb{R}^n, w)$, due to Theorem 3.7,

$$\mu_{f,w}(x, t) := |t^2 \Delta e^{-t^2 \Delta} (I - e^{-t^2 \Delta}) f(x)|^2 \frac{t^n}{w(B(x, t))} \frac{d\mu(y)dt}{t}$$

is a Carleson measure. Let $\psi_t(x - y) = t^{-n} \psi((x - y)/t)$ be the kernel of $t^2 \Delta e^{-t^2 \Delta} (I - e^{-t^2 \Delta})$. Then $\psi \in \mathcal{S}$. By Corollary 1 in [HSV], $f \in BMO(\mathbb{R}^n, w)$ and $\|f\|_{BMO(\mathbb{R}^n, w)} \leq C \|f\|_{BMO_\Delta(\mathbb{R}^n, w)}$. This completes our proof.

4 Weighted Hardy spaces associated to operators

The Hardy spaces associated to operators L with Gaussian upper bounds were studied extensively recently, for example see [ADM, DY2]. In [SY] the weighted Hardy spaces associated to operators L were studied in which the Euclidean setting played an essential role. In this section, by using the tent space approach, we obtain the atomic decomposition for Hardy spaces $H_L(X, w)$, then show that the dual space of $H_L(X, w)$ is the space $BMO_L^*(X, w)$ where L^* is the adjoint operator of L .

Set

$$\overline{\mathcal{R}(L)} = \{Lu \in L^2(X) : u \in L^2(X)\}.$$

It is known that $L^2(X) = \overline{\mathcal{R}(L)} \oplus \mathcal{N}(L)$, where $\mathcal{R}(L)$ and $\mathcal{N}(L)$ stand for the range and the kernel of L , and the sum is orthogonal.

For $w \in A_\infty$ and $0 < p < \infty$, the Hardy space $H_L(X, w)$ is defined as the completion of

$$\{f \in \overline{\mathcal{R}(L)} : \|S_L f\| \in L^1(w)\}$$

in the norm $\|f\|_{H_L(X, w)} = \|S_L f\|_{L^1(w)}$, where

$$S_L f(x) = \left(\int \int_{d(x, y) < t} |Q_{t^m} f(y)|^2 \frac{d\mu(y)}{V(y, t)} \frac{dt}{t} \right)^{1/2}.$$

By the functional calculus theory, we have

$$f(x) = 4m \int_0^\infty Q_{t^m}(Q_{t^m} f)(x) \frac{dt}{t},$$

where the integral converges strongly in L^2 .

Definition 4.1 Let $w \in A_\infty$. A function $\alpha(x)$ is called an (L, w) -molecule if

$$\alpha(x) = \int_0^\infty Q_{t^m}(a(t, \cdot))(x) \frac{dt}{t}$$

where $a(t, x)$ is an $T^1(w)$ -atom supported in the tent \widehat{B} for some ball $B \subset X$.

Let T_c^p the set of all functions $f \in T^p$ with compact support in \mathbb{R}_+^{n+1} . Consider the operator π_L initially defined on T_c^p by

$$\pi_L(f)(x) = 4m \int_0^\infty Q_{t^m}(f(\cdot, t)) \frac{dt}{t}.$$

Proposition 4.2 The operator π_L , initially defined on T_c^2 , extends to a bounded linear operator from

(a) T^p to L^p for all $1 < p < \infty$;

(b) $T^1(w)$ to $H_L(X, w)$.

Proof: (a) We refer the reader to [DY2, lemm 4.3].

(b) Let $f \in T_c^1(w)$. Then by Theorem 2.5, Lemma 2.7 there exist a sequence of number $\{\lambda_i\}$ and the sequence of $T^1(w)$ -atoms such that

$$f = \sum_j \lambda_j a_j$$

in T^2 with $\sum_j |\lambda_j| < \infty$, and hence by (a)

$$\pi_L f = \sum_j \lambda_j \pi_L a_j$$

in $L^2(X)$.

This together with the L^2 -boundedness of S_L gives

$$\|S_L(\pi_L f)\|_{L^1(w)} \leq \sum_j |\lambda_j| \|S_L(\pi_L a_j)\|_{L^1(w)}.$$

To complete the proof, it suffices to show that there exists a constant C such that for any $T^1(w)$ -atom a supported in \widehat{B} for some ball B in X , we have

$$\|S_L(\pi_L a)\|_{L^1(w)} \leq C.$$

Indeed, we have

$$\|S_L(\pi_L a)\|_{L^1(w)} \leq \|S_L(\pi_L a)\|_{L^1(4B,w)} + \|S_L(\pi_L a)\|_{L^1((4B)^c,w)}.$$

For the first term, using Hölder inequality and $w \in RH_2$ we have

$$\begin{aligned} \|S_L(\pi_L a)\|_{L^1(4B,w)} &\leq \|S_L(\pi_L a)\|_{L^2} \left(\int_{4B} w(x)^2 dx \right)^{1/2} \leq c \|\pi_L a\|_{L^2} \frac{w(B)}{V(B)^{1/2}} \\ &\leq c \|a\|_{T^2} \frac{w(B)}{V(B)^{1/2}} = c \left\| \frac{w(B)}{V(B)^{1/2}} a \right\|_{T^2} = c \end{aligned}$$

where in the last inequality we use the fact that if a is an $T^1(w)$ -atom then $\frac{w(B)}{V(B)^{1/2}} a$ is an T^2 -atom.

To estimate the second term, we exploit the argument in [DY2] into our situation. For $x \notin 4B$, we have

$$\begin{aligned} (S_L(\pi_L a))^2 &= \int_0^\infty \int_{d(x,y)<t} \left| \int_0^{r_B} Q_{t^m} Q_{s^m} a(\cdot, s)(y) \frac{ds}{s} \right|^2 \frac{d\mu(y)}{V(y,t)} \frac{dt}{t} \\ &= \int_0^\infty \int_{d(x,y)<t} \left| \int_0^{r_B} \frac{s^m t^m}{(s^m + t^m)^2} (s^m + t^m)^2 L^2 e^{-(s^m + t^m)L} a(\cdot, s)(y) \frac{ds}{s} \right|^2 \frac{d\mu(y)}{V(y,t)} \frac{dt}{t} \\ &= \int_0^{r_B} \dots + \int_{r_B}^\infty \dots := I_1 + I_2. \end{aligned}$$

Let us estimate I_2 first. We have

$$I_2 \leq c \int_{r_B}^\infty \int_{d(x,y)<t} \left| \int_0^{r_B} \int_B \frac{s^m t^m}{(s^m + t^m)^2} \frac{1}{(s+t)^n} \exp\left(-c \frac{s+t+d(y,z)}{s+t}\right) a(z,s) dz \frac{ds}{s} \right|^2 \frac{d\mu(y)}{V(y,t)} \frac{dt}{t}.$$

For $t \geq r_B$, it can be verified that

$$s+t+d(y,z) \geq \frac{1}{8}(t+d(x,x_B)).$$

Therefore, we can get

$$\begin{aligned} I_2 &\leq c \int_{r_B}^\infty \int_{d(x,y)<t} \left| \int_0^{r_B} \int_B \frac{s^m t^m}{(s^m + t^m)^2} \frac{1}{(s+t)^n} \left(\frac{s+t}{t+d(x,x_B)} \right)^{n+1} a(z,s) dz \frac{ds}{s} \right|^2 \frac{d\mu(y)}{V(y,t)} \frac{dt}{t} \\ &\leq c \int_{r_B}^\infty \left| \int_0^{r_B} \int_B \frac{s^m t^m}{(s^m + t^m)^2} \frac{s+t}{(t+d(x,x_B))^{n+1}} a(z,s) dz \frac{ds}{s} \right|^2 \frac{dt}{t}. \end{aligned}$$

Hence

$$\begin{aligned}
I_2 &\leq c \int_{r_B}^{\infty} \left| \int_0^{r_B} \int_B \frac{s^2 t^{-1}}{(t + d(x, x_B))^{n+1}} a(z, s) dz \frac{ds}{s} \right|^2 \frac{dt}{t} \\
&\leq c \int_{r_B}^{\infty} \left| \int_0^{r_B} \frac{s^2 t^{-1}}{(t + d(x, x_B))^{n+1}} V(B)^{1/2} \|a(\cdot, s)\|_{L^2(B)} \frac{ds}{s} \right|^2 \frac{dt}{t} \\
&\leq c \int_{r_B}^{\infty} \left(\int_0^{r_B} \frac{s^4 t^{-2}}{(t + d(x, x_B))^{2n+2}} \frac{ds}{s} \times \int_0^{r_B} V(B) \|a(\cdot, s)\|_{L^2(B)}^2 \frac{ds}{s} \right) \frac{dt}{t}
\end{aligned}$$

Since a is an $T^1(w)$ atom, we have

$$\int_0^{r_B} \|a(\cdot, s)\|_{L^2(B)}^2 \frac{ds}{s} \leq \frac{V(B)^2}{w(B)}.$$

This implies

$$\begin{aligned}
I_2 &\leq c \int_{r_B}^{\infty} \frac{r_B^4 t^{-2}}{(t + d(x, x_B))^{2n+2}} \frac{V(B)^2}{w(B)^2} \frac{dt}{t} \\
&\leq c \frac{r_B^2}{(r_B + d(x, x_B))^{2n+2}} \frac{V(B)^2}{w(B)^2}.
\end{aligned}$$

A similar argument gives

$$I_1 \leq c \frac{r_B^2}{(r_B + d(x, x_B))^{2n+2}} \frac{V(B)^2}{w(B)^2}.$$

Using the estimates I_1, I_2 and $w \in A_1 \cap RH_2$, we have

$$\begin{aligned}
\|S_L(\pi_L a)\|_{L^1((4B)^c, w)} &= \sum_{j \geq 3} \|S_L(\pi_L a)\|_{L^1(S_j(B), w)} \\
&\leq \sum_{j \geq 3} \|S_L(\pi_L a)\|_{L^2(S_j(B))} \left(\int_{2^j B} w(x)^2 dx \right)^{1/2} \\
&\leq \sum_{j \geq 3} \frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)^{1/2}} \left(\int_{S_j(B)} \frac{r_B^2}{(r_B + d(x, x_B))^{2n+2}} dx \right)^{1/2} \\
&\leq \sum_{j \geq 3} \frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)^{1/2}} \frac{2^{-j}}{V(2^j B)^{1/2}} \\
&\leq C.
\end{aligned}$$

This completes the proof.

We have the following estimate by using a similar argument to the proof above.

Lemma 4.3 *Let $M \in \mathbb{N}$. For any L^2 -function f support on a ball B , there exists a constant c such that*

$$\|(I - P_{r_B^M})^M f\|_{H_L(X, w)} \leq \frac{w(B)}{V(B)^{1/2}} \|f\|_{L^2}.$$

Theorem 4.4 *Assume that (X, μ) satisfies the following condition: $\mu(B(x, r)) \approx r^n$ for all $x \in X$ and $r > 0$. Then the following holds:*

(i) *Let $w \in A_1$ and $f \in H_L(X, w) \cap L^2(X)$. There exist a sequence number $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence of (L, w) -molecules $\{\alpha_j\}_{j=1}^{\infty}$ such that*

$$f(x) = \sum_{j=1}^{\infty} \lambda_j \alpha_j \tag{20}$$

in both $H_L(X, w)$ and $L^2(X)$, and $\sum_{j=1}^{\infty} |\lambda_j| \leq c \|f\|_{H_L(X, w)}$.

(ii) Conversely, if $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, then for any a sequence of L -molecules $\{\alpha_j\}_{j=1}^{\infty}$ the decomposition (20) satisfies

$$\left\| \sum_{j=1}^{\infty} \lambda_j \alpha_j \right\|_{H_L(X, w)} \leq c \sum_{j=1}^{\infty} |\lambda_j|$$

for $w \in A_1 \cap RH_2$.

Proof: (i) Since $f \in H_L(X, w) \cap L^2(X)$, $Q_{t^m} f \in T^1(w) \cap T^2$. Therefore, by Theorem 2.5, there exist a sequence number $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence of $T^1(w)$ -atoms $\{a_j\}_{j=1}^{\infty}$ such that

$$Q_{t^m} f = \sum_{j=1}^{\infty} \lambda_j a_j$$

in both $T^1(w)$ and T^2 . Due to Proposition 4.2, we have

$$f = c \sum_{j=1}^{\infty} \lambda_j \pi_L a_j = c \sum_{j=1}^{\infty} \lambda_j \alpha_j$$

in both $H_L(X, w)$ and $L^2(X)$.

(ii) Part (ii) is a direct consequence of (ii) Proposition 4.2.

The main result of this section is the following theorem.

Theorem 4.5 *Let $M \in \mathbb{N}$. For $w \in A_1 \cap RH_2$, the dual space of $H_L(X, w)$ is the $BMO_{L^*, M}(X, w)$ space, in the following sense*

(i) *If $f \in BMO_{L^*, M}(X, w)$ then the linear function l given by*

$$l(g) = \int_X f(x) g(x) d\mu(x), \quad (21)$$

initially defined on the dense subspace $H_L(X, w) \cap L^2(X)$, has a unique extension to $H_L(X, w)$.

(ii) *Conversely, every continuous linear functional l on the $H_L(X, w)$ space can be realized as above; i.e., there exists $f \in BMO_{L^*, M}(X, w)$ such that (21) holds and $\|f\|_{BMO_{L^*, M}(X, w)} \leq c \|l\|$.*

The following identity related to Carleson measure will plays an important role in the proof of Theorem 4.5.

Suppose that $f \in \mathcal{M}$ such that $\mu_{f, w}(x, t) = |Q_{t^m}^*(I - P_{t^m}^*)^M f(x)| \frac{V(x, t)}{w(B(x, t))} \frac{d\mu(x) dt}{t}$ is a w -Carleson measure and g is an (L, w) -molecule of $H_L(X, w)$. Let

$$F(x, t) = Q_{t^m}^*(I - P_{t^m}^*)^M f(x) \text{ and } G(x, t) = Q_{t^m} g(x), \quad (x, t) \in X \times (0, \infty). \quad (22)$$

Then the following identity holds.

Proposition 4.6 *For any functions F and G defined as in (22), we have the following identity with some constant b_M :*

$$\int_X f(x) g(x) d\mu(x) = b_M \int_{X \times (0, \infty)} F(x, t) G(x, t) \frac{d\mu(x) dt}{t}. \quad (23)$$

As a consequence, if $f \in BMO_{L^, M}(X, w)$ and $h \in H_L(X, w) \cap L^2(X)$, the above equality (23) holds.*

The proof is analogous to [DY2, Proposition 5.1] with some minor modifications. We omit details here.

Proof of Theorem 4.5: It can be verified that $H_L(X, w) \cap L^2(X)$ is dense in $H_L(X, w)$. For any $f \in H_L(X, w) \cap L^2(X)$ and $g \in BMO_{L^*, M}(X, w)$, due to Proposition 4.6 and Theorem 2.6, we obtain

$$\begin{aligned} \left| \int_X f(x)g(x)dx \right| &= b_m \int_{X \times (0, \infty)} Q_{t^m}^*(I - P_{t^m}^*)^M f(x) Q_{t^m} g(x) \frac{d\mu(x)dt}{t} \\ &\leq \| \mathcal{C}(Q_{t^m}^*(I - P_{t^m}^*)^M f) \|_{L^\infty} \| \mathcal{A}(Q_{t^m} g) \|_{L^1(w)} \\ &\leq c \| f \|_{BMO_{L^*, M}(X, w)} \| g \|_{H_L(X, w)}. \end{aligned}$$

This completes (i).

(ii) We will adapt the argument in Theorem 3.1 of [DY2] to our present situation. We define

$$\Omega_L := \{h(x, t) : h(x, t) = Q_t g(x) \text{ for some } g \in H_L(X, w)\}.$$

This yields that $\Omega_L \subset T^1(w)$. By (b) of Proposition 4.2, we have

$$\mathcal{R}(h)(x) = 4m \int_0^\infty Q_{t^m}(h_t)(x) \frac{dt}{t} \in H_L(X, w)$$

for $h_t(x) \in T^1(w)$.

On the other hand, for any $f \in H_L(X, w) \cap L^2(X)$,

$$g(x) = 4m \int_0^\infty Q_{t^m} Q_{t^m} g(x) \frac{dt}{t}.$$

It therefore follows that for each continuous linear function l on $H_L(X, w)$, we obtain for $g \in H_L(X, w) \cap L^2(X)$,

$$l(g) = l \circ \mathcal{R} \circ Q_{t^m}(g).$$

Since $l \circ \mathcal{R}$ is continuous linear function on Ω_L which satisfies

$$\|l \circ \mathcal{R}\|_{T_2^1 \rightarrow \mathbb{C}} \leq \|l\|_{(H_L(X, w))^*} \|\mathcal{R}\|_{T^1(w) \rightarrow H_L(X, w)} \leq c.$$

The Hahn-Banach Theorem tells us that we can extend $l \circ \mathcal{R}$ to a bounded linear function on $T^1(w)$. By Theorem 2.6, there exists $u(x, t) \in T^\infty(w)$ such that

$$\begin{aligned} l(g) &= l \circ \mathcal{R} \circ Q_{t^m}(g) = \int_{\mathbb{R}_+^{n+1}} u(x, t) Q_{t^m} g(x) \frac{d\mu(x)dt}{t} \\ &= \int_X \left(\int_0^\infty Q_{t^m}^* u(x, t) \frac{dt}{t} \right) g(x) dx = \int_X f(x) g(x) d\mu(x) \end{aligned}$$

where $f(x) = \int_0^\infty Q_{t^m}^* u(x, t) \frac{dt}{t}$.

We now prove that $f \in BMO_{L^*, M}(X, w)$. By Theorem 3.5 we have, for any ball $B \subset X$ and $w \in A_1 \cap RH_2$,

$$\begin{aligned} \left(\frac{1}{w(B)} \int_B |(I - P_{r_B^m}^*)^M f(x)|^2 w^{-1}(x) d\mu(x) \right)^{1/2} &\leq \left(\frac{V(B)}{w(B)^2} \int_B |(I - P_{r_B^m}^*)^M f(x)|^2 d\mu(x) \right)^{1/2} \\ &\leq \frac{V(B)^{1/2}}{w(B)} \sup_{\|g\|_{L^2(B)}=1} \left| \int_X (I - P_{r_B^m}^*)^M f(x) g(x) d\mu(x) \right| \\ &= \frac{V(B)^{1/2}}{w(B)} \sup_{\|g\|_{L^2(B)}=1} \left| \int_X f(x) (I - P_{r_B^m}^*)^M g(x) d\mu(x) \right| \end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{1}{w(B)} \int_B |(I - P_{r_B^m}^*)^M f(x)|^2 w^{-1}(x) d\mu(x) \right)^{1/2} &\leq \frac{V(B)^{1/2}}{w(B)} \|l\| \sup_{\|g\|_{L^2(B)}=1} \|(I - P_{r_B^m})g(x)\|_{H_L(X,w)} \\
&\leq \frac{V(B)^{1/2}}{w(B)} \|l\| \frac{w(B)}{V(B)^{1/2}} \|g\|_{L^2(B)} \\
&\leq c \|l\|.
\end{aligned}$$

This completes our proof.

5 An Interpolation Theorem

In this section, we study the interpolation of the weighted BMO space $BMO_{\mathcal{A}}(X, w)$ in general setting of spaces of homogeneous type. Firstly, We review the concept of the sharp maximal operator $M_{\mathcal{A}}^{\sharp}$ associated to the family $\{\mathcal{A}_t\}_{t>0}$ defined on $L^p(X), p > 0$ as well as its basic properties in [Ma],

$$M_{\mathcal{A}}^{\sharp} f(x) = \sup_{x \in B} \left(\frac{1}{\mu(B)} \int_B |(I - \mathcal{A}_{t_B})f(x)| d\mu(x) \right),$$

where $t_B = r_B^m$.

We recall the following results in [Ma].

Theorem 5.1 *Let $0 < p < \infty$ and $w \in A_{\infty}$. For every $f \in L_0^1(X)$ with $Mf \in L^p(w)$, we have*

- (i) $M_{\mathcal{A}}^{\sharp} f(x) \leq CMf(x)$.
- (ii) $\|Mf\|_{L^p(w)} \leq C \|M_{\mathcal{A}}^{\sharp} f\|_{L^p(w)}$ if $\mu(X) = \infty$.
- (iii) $\|Mf\|_{L^p(w)} \leq C (\|M_{\mathcal{A}}^{\sharp} f\|_{L^p(w)} + \|f\|_{L^1})$ if $\mu(X) < \infty$.

In what follows, the operator T is said to be bounded from wL^{∞} to $BMO_{\mathcal{A}}(X, w)$ if there exists c such that for all $f \in L^{\infty}(X)$,

$$\|T(fw)\|_{BMO_{\mathcal{A}}(X,w)} d\mu(x) \leq c \|f\|_{L^{\infty}}.$$

In [B], an interpolation theorem for the classical weighted BMO was given. It is interesting that our weighted $BMO_{\mathcal{A}}(X, w)$ can be considered as a good substitution the classical weighted BMO in the sense of interpolation. By adapting the arguments in [B] to our situation, we will establish an interpolation theorem concerning the our weighted BMO spaces $BMO_{\mathcal{A}}(X, w)$.

Theorem 5.2 *Assume that T is a linear operator which is bounded on $L^2(X)$. Assume also that T is bounded from wL^{∞} to $BMO_{\mathcal{A}}(X, w)$ and T^* is bounded from wL^{∞} to $BMO_{\mathcal{A}^*}(X, w)$ for all $w \in A_1 \cap RH_s$ with $1 \leq s < \infty$. Then T is bounded on $L^p(w)$ for all $s < p < \infty$, $w \in A_{p/s}$.*

Proof: For the sake of simplicity we assume that $\mu(X) = \infty$, the case that $\mu(X) < \infty$ can be treated in the same way. When $w \equiv 1$, the operator T is bounded from L^{∞} to $BMO_{\mathcal{A}}(X, w)$. Due to [DY2, Theorem 5.2], T is bounded on $L^p(X)$ for $p \in (2, \infty)$. By duality, one gets that T is bounded on $L^p(X)$ for $p \in (1, \infty)$.

Now for $w \in A_1 \cap RH_s$ and $f \in L^\infty(X)$, we have

$$\begin{aligned} w^{-1}(x)M_{\mathcal{A}}^\sharp(T^*(wf))(x) &= \sup_{B \ni x} \frac{1}{V(B)} \int_B |(I - \mathcal{A}_{t_B})T^*(wf)(y)| d\mu(y) w^{-1}(x) \\ &\leq \sup_{B \ni x} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T^*(wf)(y)| d\mu(y) \\ &\leq c\|T^*(wf)\|_{BMO_{\mathcal{A}}(X,w)} \leq c\|f\|_{L^\infty} \end{aligned}$$

for all $x \in X$. This implies that the operator $w^{-1}M_{\mathcal{A},T^*w}^\sharp$ is bounded on $L^\infty(X)$, where $M_{\mathcal{A},T^*w}^\sharp$ is defined by $M_{\mathcal{A},T^*w}^\sharp f = M_{\mathcal{A}}^\sharp(T^*(wf))$. On the other hand due to Proposition 5.1 and the L^2 -boundedness of T^* , $M_{\mathcal{A},T^*}^\sharp$ is bounded on $L^2(X)$. This together with the complex interpolation method gives

$$u^{2/p-1}M_{\mathcal{A}}^\sharp(T^*u^{1-2/q}) : L^q \rightarrow L^q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

This implies

$$M_{\mathcal{A},T^*}^\sharp : L^q(w^{2-q}) \rightarrow L^q(w^{2-q}).$$

Using Theorem 5.1, we have

$$T^* : L^q(w^{2-q}) \rightarrow L^q(w^{2-q}).$$

Let $g \in L^q(w^{2-q})$ and $f \in L^p(w^{2-p})$. We have

$$\int_X |(Tf)g| d\mu = \int_X |fw^{1-2/q}(T^*g)w^{2/q-1}| d\mu \leq \|T^*g\|_{L^q(w^{2-q})} \|f\|_{L^p(w^{2-p})}.$$

By duality, $T : L^p(w^{2-p}) \rightarrow L^p(w^{2-p})$, or, $w^{2/p-1}Tw^{1-2/p} : L^p \rightarrow L^p$.

On the other hand, for $f \in L^p$ and $g \in L^q$, we have

$$\int_X |T(fw^{2/q-1})w^{1-2/q}g| d\mu = \int_X |f \times w^{2/q-1}T^*(w^{1-2/q}g)| d\mu \leq c\|f\|_{L^p} \|g\|_{L^q},$$

and hence $w^{1-2/q}Tw^{2/q-1} : L^p \rightarrow L^p$.

Since we can interchange T and T^* , we can show that for $\frac{1}{p} + \frac{1}{q} = 1$, p near 1, and $w, v \in A_1$,

$$w^{1-2/q}Tw^{2/q-1} : L^p \rightarrow L^p \text{ and } v^{2/q-1}Tv^{1-2/q} : L^q \rightarrow L^q.$$

By interpolation, we obtain

$$w^{\alpha(t)}v^{\beta(t)}T(w^{-\alpha(t)}v^{-\beta(t)}) : L^{1/t} \rightarrow L^{1/t} \text{ for } \frac{1}{q} \leq t \leq \frac{1}{p}$$

for all $v, w \in A_1 \cap RH_s$, where $\alpha(t) = t - \frac{1}{q}$ and $\beta(t) = t - \frac{1}{p}$.

This gives $T : L^{p_0}(u) \rightarrow L^{p_0}(u)$ whenever

$$u = w^{p_0\alpha(1/p_0)}v^{p_0\beta(1/p_0)}, \quad w, v \in A_1 \cap RH_s \text{ and } p < p_0 < q. \quad (24)$$

Take $p_0 = (q + s) - qs/p$ and $r_0 = \frac{pq}{q-p}$. For any $u \in A_{p_0/s}$, by Jones Factorization, there exist $u_1, u_2 \in A_1$ such that $u = u_1u_2^{1-p_0/s}$, see [J]. Setting $u_1 = w_1^s$ and $u_2 = w_2^s$, then $w_1, w_2 \in A_1 \cap RH_s$. Hence, we can pick $\delta > 0$ so that $u_1^{1+\delta}, u_2^{1+\delta} \in A_1$. For p close enough to 1, $r_0 < 1 + \delta$ and hence $u_1^{r_0} = w_1^{r_0s}, u_2^{r_0} = w_2^{r_0s} \in A_1$. This implies $w_1^{r_0}, w_2^{r_0} \in A_1 \cap RH_s$. Due to (24), T is bounded on $L^{p_0}(v)$, here $v = (w_1^{r_0})^{p_0\alpha(1/p_0)}(w_2^{r_0})^{p_0\beta(1/p_0)} = w_1^s w_2^{s(1-p_0)} = u$. Applying Theorem 4.9 in [AM1], T is bounded on $L^p(w)$ for all $s < p < \infty$ and $w \in A_{p/s}$. This completes our proof.

6 Applications to boundedness of singular integrals

Let X be a space of homogeneous type (X, d, μ) . Let T be a bounded linear operator from $L^2(X)$ to $L^2(X)$ with kernel k such that for every f in $L^\infty(X)$ with bounded support,

$$Tf(x) = \int_X k(x, y)f(y)d\mu(y),$$

for μ -almost all $x \notin \text{supp} f$. We will consider the following conditions:

(H1) There exists a class of approximation to the identity $\{\mathcal{A}_t\}_{t>0}$ satisfying (3) such that the operators $(T - \mathcal{A}_t T)$ and $(T - T\mathcal{A}_t)$ have associated kernels $K_t^1(x, y)$ and $K_t^2(x, y)$ respectively and there exist positive constants α and c_1, c_2 such that

$$\max\{|K_t^2(x, y)|, |K_t^1(x, y)|\} \leq c_2 \frac{1}{V(x, d(x, y))} \frac{t^{\alpha/m}}{d(x, y)^\alpha}$$

when $d(x, y) \geq c_1 t^{1/m}$.

(H2) There exists a class of approximation to the identity $\{\mathcal{A}_t\}_{t>0}$ satisfying (3) such that the operators $(T - \mathcal{A}_t T)$ and $(T - T\mathcal{A}_t)$ have associated kernels $K_t^1(x, y)$ and $K_t^2(x, y)$ so that there exist $1 < p_0 < \infty$ and $\delta > n/p'_0$ such that for any ball $B \subset X$ we have

$$\left(\int_{S_j(B)} |K_{r_B^m}(z, y)|^{p_0} d\mu(y) \right)^{1/p_0} \leq C 2^{-j\delta} V(B)^{1/p_0-1} \quad (25)$$

for all $z \in B$ and all $j \geq 2$, where $K_t(y, z)$ is either $K_t^1(y, z)$ or $K_t^2(y, z)$.

It was proven in [DM] that if T is an operator satisfying (H1) or (H2) above, then T bounded on $L^p(X)$ for $1 < p < 2$. Note that condition (H2) does not require the regularity assumption on space variables. This allows us to obtain L^p -boundedness of certain singular integrals with nonsmooth kernels such as the holomorphic functional calculi and spectral multipliers of L , see [DM, DOS]. Note that it was proved in [DM] that the holomorphic functional calculi $f(L)$ satisfies (H1). Meanwhile the spectral multipliers of L satisfy the estimate (H2), see the proof of Theorem 6.3 below.

We now prove the following theorems:

Theorem 6.1 *Let T be an operator satisfying (H1). Then for any $w \in A_1$, T and T^* are bounded from $wL^\infty(X)$ to $BMO_{\mathcal{A}}(X, w)$ and from $wL^\infty(X)$ to $BMO_{\mathcal{A}^*}(X)$. Then, by interpolation, T is bounded on $L^p(w)$ for all $p \in (1, \infty)$ and $w \in A_p$.*

Proof: For $f \in L^\infty$, we claim that

$$\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(fw)(x)| d\mu(x) \leq C \|f\|_{L^\infty}$$

for any ball $B \subset X$.

Set $f = f_1 + f_2$ where $f_1 = f\chi_{cB}$ with $c = \max\{c_1, 4\}$. We have

$$\begin{aligned} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(fw)(x)| d\mu(x) &\leq \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_1w)(x)| d\mu(x) \\ &\quad + \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_2w)(x)| d\mu(x) \\ &= I_1 + I_2. \end{aligned}$$

Let us estimate I_1 first. Since $w \in A_1$ then there exists $r > 1$ such that $w \in RH_r$. Using the L^p boundedness of T and the Hardy-Littlewood maximal function, we have

$$\begin{aligned}
I_1 &\leq c \frac{1}{w(B)} \int_B M(T(f_1 w))(x) d\mu(x) \\
&\leq c \frac{1}{w(B)} \|T(f_1 w)\|_{L^r} V(B)^{1/r'} \\
&\leq c \|f\|_{L^\infty} \frac{1}{w(B)} \left(\int_{cB} w^r(x) d\mu(x) \right)^{1/r} V(B)^{1/r'} \\
&\leq c \|f\|_{L^\infty} \frac{1}{w(B)} \frac{w(B)}{V(B)} V(B)^{1/r} V(B)^{1/r'} = c \|f\|_{L^\infty}.
\end{aligned}$$

For the second term, by (b) we have

$$\begin{aligned}
I_2 &\leq \frac{1}{w(B)} \int_B \int_{X \setminus cB} K_{t_B}^1(x, y) (f_2 w)(y) d\mu(y) d\mu(x) \\
&\leq \frac{1}{w(B)} \int_B \int_{X \setminus cB} \frac{1}{V(x, d(x, y))} \frac{r_B^\alpha}{d(x, y)^\alpha} (f_2 w)(y) d\mu(y) d\mu(x) \\
&\leq c \|f\|_{L^\infty} \frac{1}{w(B)} \int_B \int_{X \setminus cB} \frac{1}{V(x, d(x, y))} \frac{r_B^\alpha}{d(x, y)^\alpha} w(y) d\mu(y) d\mu(x).
\end{aligned}$$

Since $c > 4$, we have

$$\begin{aligned}
I_2 &\leq c \|f\|_{L^\infty} \sum_{j \geq 2} \frac{1}{w(B)} \int_B \int_{S_j(B)} \frac{1}{V(x, d(x, y))} \frac{r_B^\alpha}{d(x, y)^\alpha} w(y) d\mu(y) d\mu(x) \\
&\leq c \|f\|_{L^\infty} \sum_{j \geq 2} 2^{-j\alpha} \frac{V(B)}{w(B)} \int_B \frac{w(2^j B)}{V(2^j B)} \\
&\leq c \|f\|_{L^\infty}.
\end{aligned}$$

The boundedness of T^* can be treated similarly. This completes our proof.

Theorem 6.2 *Let T be an operator satisfying (H2). Then for any $w \in A_1 \cap RH_{p'_0}$, T and T^* are bounded from $wL^\infty(X)$ to $BMO_{\mathcal{A}}(X, w)$ and from $wL^\infty(X)$ to $BMO_{\mathcal{A}^*}(X)$. Then, by interpolation, T is bounded on $L^p(w)$ for all $p \in (p'_0, \infty)$ and $w \in A_p/p'_0$.*

Proof: For $f \in L^\infty$ and $w \in A_1 \cap RH_{p'_0}$, we will claim that

$$\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(fw)(x)| d\mu(x) \leq C \|f\|_{L^\infty}$$

for balls $B \subset X$.

Using the decomposition $f = \sum_{j \geq 2} f_j + f_0$ where $f_0 = f \chi_{2B}$ and $f_j = f \chi_{S_j(B)}$, We have

$$\begin{aligned}
\frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(fw)(x)| d\mu(x) &\leq \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_0 w)(x)| d\mu(x) \\
&\quad + \sum_{j \geq 2} \frac{1}{w(B)} \int_B |(I - \mathcal{A}_{t_B})T(f_j w)(x)| d\mu(x) \\
&= I_0 + \sum_{j \geq 2} I_j.
\end{aligned}$$

Since $w \in RH_{p'_0}$, using the L^p boundedness of T and the Hardy-Littlewood maximal function, we have

$$\begin{aligned}
I_0 &\leq c \frac{1}{w(B)} \int_B M(T(f_0 w))(x) d\mu(x) \\
&\leq c \frac{1}{w(B)} \|T(f_1 w)\|_{L^{p'_0}} V(B)^{1/p_0} \\
&\leq c \|f\|_{L^\infty} \frac{1}{w(B)} \left(\int_{2B} w^{p'_0}(x) d\mu(x) \right)^{1/p'_0} V(B)^{1/p_0} \\
&\leq c \|f\|_{L^\infty} \frac{1}{w(B)} \frac{w(B)}{V(B)} V(B)^{1/p_0} V(B)^{1/p'_0} = c \|f\|_{L^\infty}.
\end{aligned}$$

For $j \geq 2$, by (H3) and Hölder inequality, we have

$$\begin{aligned}
I_j &\leq \frac{1}{w(B)} \int_B \int_{S_j(B)} |K_{t_B}^1(x, y)(f_j w)(y)| d\mu(y) d\mu(x) \\
&\leq \frac{1}{w(B)} \int_B \left(\int_{S_j(B)} |K_{t_B}^1(x, y)|^{p_0} d\mu(y) \right)^{1/p_0} \left(\int_{S_j(B)} |f_j(y)w(y)|^{p'_0} d\mu(y) \right)^{1/p'_0} d\mu(x) \\
&\leq C \frac{V(B)}{w(B)} 2^{-j\delta} V(B)^{1/p_0-1} \|f\|_{L^\infty} \left(\int_{2^j B} |w(y)|^{p'_0} d\mu(y) \right)^{1/p'_0} d\mu(x) \\
&\leq C \frac{V(B)}{w(B)} 2^{-j\delta} V(B)^{1/p_0-1} \|f\|_{L^\infty} \frac{w(2^j B)}{V(2^j B)} V(2^j B)^{1/p'_0} \\
&\leq C 2^{-j(\delta-n/p'_0)} \frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)} \|f\|_{L^\infty}.
\end{aligned}$$

Since $w \in A_1$,

$$\frac{V(B)}{w(B)} \frac{w(2^j B)}{V(2^j B)} \leq C$$

Therefore, $I_j \leq C 2^{-j(\delta-n/p'_0)} \|f\|_{L^\infty}$ and hence $\sum_{j \geq 2} I_j \leq C \|f\|_{L^\infty}$ provided $\delta > n/p'_0$. This yields that T is bounded from $wL^\infty(X)$ to $BMO_{\mathcal{A}}(X, w)$. The boundedness of T^* can be treated similarly. This completes our proof.

We end this section by considering the boundedness of spectral multipliers.

Let L be a non-negative self-adjoint operator on $L^2(X)$ and the operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ whose kernels $p_t(x, y)$ satisfies Gaussian upper bound (15).

By the spectral theorem, for any bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$, one can define the operator

$$F(L) = \int_0^\infty F(\lambda) dE(\lambda) \quad (26)$$

which is bounded on $L^2(X)$. We have the following result.

Theorem 6.3 *Suppose that $n > s > n/2$ and for any $R > 0$ and all Borel functions F such that $\text{supp} F \subset [0, R]$,*

$$\int_X |K_{F(\sqrt{L})}(x, y)|^2 d\mu(x) \leq \frac{C}{V(y, R^{-1})} \|\delta_R F\|_{L^q}^2 \quad (27)$$

for some $q \in [2, \infty]$. Then for any Borel function F such that

$$\sup_{t>0} \|\eta \delta_t F\|_{W_s^q} < \infty,$$

where $\delta_t F(\lambda) = F(t\lambda)$, $\|F\|_{W_s^q} = \|(I - d^2/dx^2)^{s/2} F\|_{L^q}$, the operator $F(L)$ and $F(L)^* = \overline{F}(L)$ is bounded from wL^∞ to $BMO_{\mathcal{A}}(X, w)$ for all $w \in A_1 \cap RH_{r'_0}$, where $\mathcal{A}_t = I - (I - e^{-tL})^M$ for $M > \frac{s}{m}$ and $r_0 = n/s$. Hence by interpolation, $F(L)$ is bounded on $L^p(w)$ for $w \in A_p/r_0$ and $p \in (r_0, \infty)$.

Proof: From the proof of Theorem 4.5 in [AD1], we get that (H3) holds for $T := F(L)$ and the family $\mathcal{A}_t := I - (I - e^{-tL})^M$ for $M > \frac{s}{m}$ and $p_0 := r'_0$. Hence, as a direct consequence of Theorem 6.2, the proof is complete.

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